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"Analysis of Fractional Order Mathematical Models Using Various Techniques and Mathematical Softwares "

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1 Introduction to Proposed Research Topic:

This Synopsis provides a brief overview of the literature review in preparation for the research work. Ordinary differential equations and integration are generalised to arbitrary non integer orders in fractional differential equations. It is frequently and effectively used to describe a wide range of engineering, physics, economics, and science phenomena. Mathematical models can be studied more precisely using fractional derivative formulations. Researchers and mathematicians have recently been interested in the topic of mathematical modelling. Mathematical models are used to solve many real-life problems like launching a satellite, controlling pollution due to vehicles etc.

Fractional order calculus has been researched as an alternative calculus in mathematics in recent years, and it has been used in a variety of scientific fields. The Adomian decomposition method, differential transform method, etc., has been used to solve modelling problems. Fractional order model are more adequate than integer order model.

Mathematical modelling involves different steps for solving the model. Initially problem statement is taken and write it as in an equivalent mathematical formulation. Then solve the mathematical problem and interpret its solution by various methods. It can be seen that obtained solution is valid or not. So, the stages involved in mathematical modelling are formulation, solution, interpretation and validation.

Many mathematical softwares are available like MATLAB, Python etc for research. Some of these softwares enable to achieve high level of logical-analytical solutions by viewing graphical solution of problem. With the help of mathematical softwares, the graph of solution can be obtained and mathematical softwares make the research easier for comparing different graphs of solution.

2 Statement of Proposed Problem:

Numerous problems in Physics, Mathematics, Chemistery, Engineering can be modelled using modelling methods. Different methods can be used for solving different models. Analysis of different methods for the study of designed model. Stability of models using different methods like FADM, FDTM etc can be checked. Try to analyze the effectiveness of methods used for different models. Study the effect of two or more methods simultaneously used for a model and check the stability of model, convergence of model.

The solution of model can be analyze numerically and graphically by using mathematical softwares such as MATLAB, Psilab, Python etc. Comparison of solution of model solved with the help of integer order method and solution of model obtained by fractional order method can be done.With the help of Mathematical softwares, will try to plot the graph of solution of the model using various methods and try to compare the graphs.

3 Review of literature and development in the research area

The fractional mathematical system modeling for the phenomena of vaccinated hepatitis B has been investigated by some contraction theorems of closed norm space for qualitative analysis which can be found in [1]. Solution of space fractional diffusion equation [2], determined by ADM method are very good. A software was developed using one-dimensional non-linear equations of blood flow which was solved using the Riemann based methods constructed within the finite volume framework [3]. The software package can assist clinicians in early screening of patients at risk. Salma M. Al-Tuwairqi [4] proposed a mathematical model to predict the future of Snapchat. The motivation of this work came from a similar study that was conducted for Facebook [5]. The difference between our Snapchat model and the Facebook model is that we included a new class

which represents the famous individuals within the population. These individuals are either famous and use Snapchat or became famous after being active users of Snapchat. We aimed to investigate the role of famous class in the growth of Snapchat. The mathematical model was analyzed qualitatively and numerically. In [6], from the discharge results of lead acid battery 12 V 20 AH, mathematical model and the experiment have the similarity is shown.

Variational iteration Method (VIM) is first proposed by Ji-Huan He [7]. In the year 1998, to solve fractional differential equations with great success following the above idea applied VIM to more complex fractional differential equations, showing the effectiveness and accuracy of the method. A numerical method based on the Adomian Decomposition Method (ADM) which has been used from the 1970's to the 1990's by George Adomain [8], The Differential Transform Method(DTM) has been successfully developed by Zhou [9] in electric circuit analysis. DTM has been used to solve linear and non-linear differential equations.

Some Useful Definitions and Methods

Riemann Liouville Fractional Derivative :

Riemann Liouville α fractional derivative [10] of f, for $\alpha \in [n-1, n)$, is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{(\alpha-n+1)}} dx$$

Caputo's Fractional Derivative :

Caputo's α Fractional Derivative [10] of f, for $\alpha \in [n-1, n)$, is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{(\alpha-n+1)}} dx$$

Conformable Fractional Derivative :

Let f :$[0, \infty) \to$ R be a function. Then, the conformable fractional derivative [11] of f of order $\alpha, 0 < \alpha \leq 1$, is defined by,

$$T_{\alpha}\{f(t)\} = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}, \quad t > 0$$

Every real function that satisfied above equation and corresponding limit that exists is

called the α-differentiable function.

Some Laplace Transform methods are discussed as follows

1)Laplace transform of Riemann Liouville Fractional Derivative Method

The Laplace Transform of Riemann Liouville Fractional derivative [10] is given by $L\left\{{}_0D_t^p f(t);s\right\} = s^pF(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{p-k-1} f(t)\right]_{t=0}, \qquad (n-1 \le p < n).$

Laplace transform of Caputo Fractional Derivative Method

The Laplace Transform of Caputo Fractional derivative [10] is given by

$$L\left\{{}_0^cD_t^p f(t)\right\} = s^pF(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$

$$(n-1 < p \le n), p > 0$$

2) Fractional Differential Transform Method

The Fractional Diffrential Transform [12] used to find an analytical solution of linear and nonlinear fractional differential equations.

Let the fractional power series of an analytical and continuous function $\varphi(\zeta)$ in Riemann-Liouville sense is as follows [**?**]:

$$\varphi(\zeta) = \sum_{k=0}^{\infty} \Phi(k) \left(\zeta - \zeta_0\right)^{k/\alpha}, \tag{2.1.1}$$

where α and $\Phi(k)$ are the order of fraction and FDT of $\varphi(\zeta)$ respectively. Let fractional IVP, in terms of the Caputo sense are as fallows.

$$\Phi(k) = \begin{cases} \text{If } k/\alpha \in Z^+, \frac{1}{(k/\alpha)!}\left[\frac{\mathrm{d}^{k/\alpha}\varphi(\zeta)}{\mathrm{d}\zeta^{k/\alpha}}\right]_{\zeta=\zeta_0} & \text{for } k = 0,1,2,\ldots,(q\alpha-1) \\ \text{If } k/\alpha \notin Z^+ & 0, \end{cases} \tag{2.1.2}$$

where, q denotes the order of the fractional differential equation under consideration.

3) Fractional Adomin Decomposition Method :

Fractional Adomin Decomposition Method is discussed in [8].

Now, see the fractional differential equation,

$$D^\alpha y(\zeta) = A(\zeta) + B(\zeta)y + C(\zeta)y^2, \quad \zeta > 0, n-1 < \alpha \le n \tag{3.1.1}$$

After applying I^α to the equation 3.1.1, we obtain,

$$y = \sum_{k=0}^{n-1} c_k \frac{\zeta^k}{k!} + I^\alpha[A(\zeta) + B(\zeta)y + C(\zeta)y^2], \quad 1 \leqslant i \leqslant n. \tag{3.1.2}$$

We adopt ADM to solve the equation 3.1.1. Let

$$y = \sum_{m=0}^{\infty} y_m(\zeta), \tag{3.1.3}$$

and

$$N(y) = \sum_{m=0}^{\infty} A_m, \tag{3.1.4}$$

where A_m are the Adomian polynomials. By using equations. 3.1.3 and 3.1.4, the equation 3.1.2, can be written as,

$$\sum_{m=0}^{\infty} y_m = \sum_{k=0}^{n-1} c_k \frac{\zeta^k}{k!} + I^\alpha \sum_{m=0}^{\infty} \left[A(\zeta) + B(\zeta) \sum_{m=0}^{\infty} y_m + C(\zeta) \sum_{m=0}^{\infty} A_m \right]. \tag{3.1.5}$$

This can be expressed as

$$\begin{aligned} y_0(\zeta) &= \sum_{k=0}^{n-1} c_k \frac{\zeta^k}{k!} + I^\alpha(A(\zeta)), \\ y_{m+1}(\zeta) &= I^\alpha(B(\zeta)y_n + C(\zeta)A_m), m \geq 0 \end{aligned} \tag{3.1.6}$$

The shortened series can be used to approximate the answer y_i.

$$\varphi_k = \sum_{m=0}^{k-1} y_m, \quad \lim_{k\to\infty} \varphi_k = y_i(\zeta).$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions are generally converge very rapidly. The convergence of the decomposition series has investigated by several researchers.

4) Fractional Power Series Method

Fractional Power series method is discussed in [14]. Let f is fractional power series representation at t_0 of the form

$$f(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^{n\alpha},$$

$$0 \leq m - 1 < \alpha \leq m, t_0 \leq t < t_0 + R,$$

and R is the radius of convergence of the FPS.
If $f(t) \in C\,[t_0, t_0 + R)$ and $D_t^{n\alpha} f(t) \in C(t_0, t_0 + R)$, for $n = 0, 1, 2, ...$
then the coefficients c_n will take the form of

$$c_n = \frac{D_t^{n\alpha} f(t)|_{t=t_0}}{\Gamma(n\alpha + 1)}$$

5) Fractional Variational Iteration Method

Fractional Variational Iteration Method is briefed in [7]. Variation iteration method is valid for fractional differential equation. We extend the application of variation iteration method to solve the fractional differential equation as follows:

$$D_t^{\alpha} y(t) + N\,[y(t)] = f(t), 0 < \alpha \leq 1,$$

Where N is operator with respect to $y(t)$ and $f(t)$ is a known function. According to the variational iteration method, we can construct the correction functional for above equation as follows:

$$y_{n+1}(t) = y_n(t) + l_t^{\alpha}\left[\lambda\left(D_t^{\alpha} y_n(t) + N\left(y_n(t)\right) - f(t)\right)\right]$$
$$= y_n(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, \lambda(s) * \left[D_s^{\alpha} y_n(s) - N(y_n(s) - f(s))\right] ds,$$

Where, λ is the general Lagrange multiplier.

To identify approximately Lagrange multiplier, some approximations must be made. The correction functional of above equation can be approximately expressed as

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)\left[y_n'(s) + N(y_n''(s)) - f(s)\right] ds$$

4 Scope of Research Work:

1) Various mathematical models can be analysed with the help of different methods.
2) The solutions obtained for mathematical models can be studied further using mathematical softwares.

3) Fractional differential equations have attracted a lot of attention in recent years. Fractional calculus has been used in practically every sector of science, engineering, and mathematics during the previous decade. Fractional calculus has applications in fluid dynamics, dynamical systems, physics, mathematical models, image processing, and astrophysics.

4) Mathematical models have use in many fields of science and engineering including quantitative biology, electro-chemistry, scattering theory, diffusion theory, physics and statistical probability theory and many branches of mathematical analysis.

5) Recently a lot of research work has done using different mathematical models in medical and biological diseases.

5 Significance and Relevance of the Research Work:

In physics, engineering, biology, mathematics and economics, fractional calculus modelling is becoming a valuable and important tool for analysing scientific problems. Fractional calculus, or integration and differentiation of an arbitrary or fractional order, provides additional tools that extend the descriptive capacity of calculus beyond the basic integer-order concepts of rate of change and area under the curve to solve real issues.

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann etc.

Fractional Differential modelling plays a vital role in Mathematics, Physics and Engineering. The purpose of the research to study analysis of methods like FADM, FDTM of fractional differential equations to obtain their solutions of mathematical models. The solution of fractional order mathematical models can be obtained graphically by using different mathematical softwares like MATLAB, Python etc. The results obtained by different methods can be compared graphically by using softwares.

6 Objectives of Proposed Research:

1) To study fractional order mathematical models.

2) To do analysis of different methods for differential models which are available in the review literature.

3) To study and analysis of development in results for fractional order mathematical models.

4) To analyse the result after applying two or more methods simultaneously to a mathematical model.

5) The proposed work is to prove that how mathematics is useful in other branches of science.

6) We will try to develop the new analytical results for fractional order mathatical models and try to obtain their solution. These solutions will be represented graphically by Mathematical softwares like MATLAB, Python etc.

7 Research Methodology:

The following steps will be included in the methodology.

1) To refer the relevant reference books on fractional order differential equations.

2) To do analysis of various existing mathematical models.

3) To study Mathematical Softwares like MATLAB, Python softwares.

4) To study an analytical and numerical methods for obtaining approximate solutions of fractional order mathematical models.

5) To represent Numerical solutions of fractional order mathematical models graphically using Sofwares like MATLAB, Python and discuss their nature.

8 Expected Outcome:

1) We will try to figure out how new conclusions and analytical approaches for fractional order mathematical models differs.

2) We will try to compare fractional order model and integer order model.

3) To find the approximate solutions of fractional order mathematical models by applying two or more methods simultaneously represent it graphically by different softwares like Matlab, Python etc.

4)To develop hybrid methods for solving Mathematical models.

9 Plan of Work:

First year plan:

1) To go through the reference books of fractional differential equations.

2) To read books related to mathematical models. 2) Try to collect number of reputed research papers in fractional order mathematical models.

3) Learning software like MATLAB, Python, etc.

Second year plan:

1) To do analysis of existing fractional order mathematical models.

2) To gain knowledge about reputed research journals and try to avoid from Pedatary Journals for publishing research paper in fake journal.

3) To visualize the solutions by graphs using MATLAB, Python and discuss the nature.

Third year plan:

1) To develop various methods for fractional order mathematical models.

2) To analysis of development in the different numerical methods for solving fractional order mathematical models.

3) To discuss stability and convergence of models.

4) To publish work in reputed journals.

5) Submission of completed work to concerned authority.

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